VARIATIONAL METHODS IN THE THEORY OF THE FLUIDITY OF A VISCOUS-PLASTIC MEDIUM

(VARIATSIONNYE METODY V TEORII TECHENII VIAZKO-PLASTICHESKOI SREDY)

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A model of a viscous-plastic medium was first proposed by F.N. Shwedov [1] and, independently, by Bingham [2], to describe the motion of structure-like suspension under shear conditions. Later H. Henckey [3] and A.A. Il'iushin [4] proposed a three-dimensional generalization of the Shwedov-Bingham equations of state and solved a number of problems for the case of plane flow of a viscous-plastic medium while keeping in mind the application of the results obtained to problems of the fluidity flow of metals.

A large number of works devoted to the investigation of various kinds of fluidity in a viscous-plastic medium on the basis of both the exact and approximate solutions of the equations of motion have beenpublished within the past 10-15 years. A detailed survey of the results obtained is given in [5, 6].

A characteristic peculiarity in problems of the fluidity of a viscous-plastic medium is the need to construct solutions in domains with unknown boundaries. This circumstance produces great difficulties in the construction of sufficiently general and efficient methods of investigating them. The most general approximate methods proposed in [4, 7-9], are valid only for very strict restrictions imposed on the nature of the motion of the viscous plastic medium.

Moreover, the possibility of a variational formulation of the appropriate problems has somehow dropped out of the field of view of specialists concerned with the investigation of the fluidity of a viscous-plastic medium, though, as will be shown below, this formulation has definite advantages as compared with the formulation in terms of differential equations. A variational formulation of problems of the plane fluidity of a viscous-plastic medium was first given in [4].

An attempt is made here to give a qualitative investigation of the general properties of some specific types of motion of a viscous-plastic medium on the basis of a variational formulation of these problems.

The variational principle for arbitrary slow flows of a viscous-plastic medium is formulated in section 1 and a specific form of the functionals is indicated in the case of plane flows and of flows in cylindrical pipes under the influence of a constant pressure drop. Furthermore, the nature of the flow in pipes is investigated qualitatively in section 2. It is shown that in a pipe of arbitrary cross-section there always exists at least one nucleus moving as a solid. A formulation of the problem of the motion of a body of arbitrary configuration in a plane channel is given in section 3. It is established that the perturbations caused by the moving body decrease exponentially with distance from it. It is also shown that if the yield point of the medium approaches zero, the solution of the problem for a viscous-plastic medium passes over into the solution for the corresponding problem for a viscous fluid.

The concluding section 4 contains a proof of the existence and uniqueness theorems for a broad class of functionals containing the functionals considered in the previous section as a particular case.

The results obtained necessitated a number of tedious calculations and proofs. In order to make the fundamental qualitative deductions clearer, the authors considered it worth while to relegate all the awkard proofs to an appendix at the end of the paper. Appropriate references are given in the text of the paper.

1. Variational principle for slow motions of a viscous-plastic medium

In the general case a viscous-plastic medium may be defined as a medium in which viscous flow is manifest under the condition that the intensity of the tangential shear stresses σ exceeds some quantity τ_0, called the yield point. For $\sigma < \tau_0$ the medium is in the rigid state.

On the basis of thermodynamic relationships for dissipative continuous media, an analytical definition of a viscous-plastic medium may be given in the form of a certain variational principle.

Variational principle [4, 10, 11]. For an invariant flux of energy through the boundary of a volume occupied by a viscous-plastic medium, the actual motion differs from any kinematically possible motion in that for the actual motion, the functional

$$J = \int_\omega [X_v^{\cdot}(e_{ij}) + X_p^{\cdot}(e_{ij}) - F_i v_i]\, d\omega - \int_\Gamma t_i v_i\, dS \qquad (1.1)$$

has a minimum.

In this relationship the e_{ij} are components of the shear rate-of-strain tensor, v_i and F_i are components of the velocity and external mass force fields along the coordinate axes, and t_i are projections of forces acting on the surface Γ bounding the volume ω occupied by the viscous-plastic medium.

The dissipative potentials $X_v^\cdot$ and $X_p^\cdot$ [12] for the viscous-plastic medium are

$$X_v^\cdot(e_{ij}) = \frac{\mu\gamma^2}{2}, \qquad X_p^\cdot(e_{ij}) = \tau_0\gamma \tag{1.2}$$

Here μ is the coefficient of viscosity of the medium, and γ is the intensity of the shear rate-of-strain tensor (the medium is assumed incompressible).

In the case of plane flows of a viscous-plastic medium and in the absence of external mass forces, (1.1) may be represented as

$$J(\Psi) = \int_\omega \left\{\frac{\mu}{2}[(P\Psi)^2 + (Q\Psi)^2] + \tau_0[(P\Psi)^2 + (Q\Psi)^2]^{1/2} d\omega - \right.$$
$$- \int_\Gamma \left(t_x \frac{\partial\Psi}{\partial y} - t_y \frac{\partial\Psi}{\partial x}\right) ds, \quad u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x} \tag{1.3}$$
$$P = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \qquad Q = 2\frac{\partial^2}{\partial x\,\partial y}$$

Here Ψ is the stream function, x, y a fixed Cartesian coordinate system, and u, v are components of the velocity vector of the particles of the medium along the coordinate axes.

The function Ψ should satisfy certain boundary conditions on the boundary Γ of the domain ω. The shape of the domain ω and the boundary conditions imposed on Ψ are determined by the specific peculiarities of the problem.

In general, the functional $J(\Psi)$ is not differentiable, and hence, has no appropriate Euler equation.

In fact:

$$J(\Psi + \lambda h) - J(\Psi) = \int_\omega \frac{\mu}{2}[2\lambda P\Psi Ph + 2\lambda Q\Psi Qh + \lambda^2 (Ph)^2 +$$
$$+ \lambda^2 (Qh)^2]\, d\omega + 2\tau_0 \int_\omega \frac{\lambda P\Psi Ph + \lambda Q\Psi Qh}{[(P\Psi + \lambda Ph)^2 + (Q\Psi + \lambda Qh)^2]^{1/2} + [(P\Psi)^2 + (Q\Psi)^2]^{1/2}} d\omega +$$
$$+ \int_\omega \frac{\lambda^2 (Ph)^2 + \lambda^2 (Qh)^2}{[(P\Psi + \lambda Ph)^2 + (Q\Psi + \lambda Qh)^2]^{1/2} + [(P\Psi)^2 + (Q\Psi)^2]^{1/2}} d\omega \tag{1.4}$$

The last integral is essential in the domain where $P\Psi = Q\Psi = 0$, which also denotes differentiability of the functional $J(\Psi)$.

In view of the above circumstance, the differential formulation of the problem on plane flows of a viscous-plastic medium proposed in [3, 4] is equivalent to the variational formulation only if the intensity of the tangental stresses everywhere exceeds the yield point of the medium.

In this case

$$(P\Psi)^2 + (Q\Psi)^2 > 0$$

everywhere and, according to (1.4), the functional $J(\Psi)$ becomes differentiable.

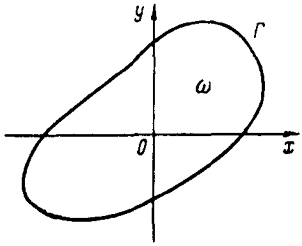

FIG. 1.

The last remark specifies the suitability of the differential equation formulation of problems on plane flows only for a very restricted class of problems. Utilization of this formulation in other cases always requires reliance on supplementary assumptions on the nature of the motion, on the shape of the domain in which the medium is in the rigid state, etc. The validity of such assumptions requires additional careful verification in each case. In addition, the question of the uniqueness of the selection of the shape of the domain where the medium is in the rigid state is still open. Examples of such an ambiguity are known for rigid-plastic models [13].

Let us now examine the steady-state motion of a viscous-plastic medium in cylindrical pipes of arbitrary cross-section subject to a given constant pressure gradient or external mass forces, for example, the force of gravity.

The trajectories of the medium's particles will be rectilinear during motion in pipes and their velocities $u(x, y)$ will be parallel to the pipe axis. In this case the functional (1.1) will be

$$J(u) = \int_{\omega} \left\{ \frac{\mu}{2} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] + \tau_0 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right]^{1/2} - Cu \right\} d\omega \qquad (1.5)$$

where the integration extends over the whole pipe cross-section (Fig. 1). Adhesion conditions are satisfied on the pipe surface so that we have for $u(x, y)$

$$u(x, y)|_{\Gamma} = 0 \qquad (1.6)$$

It may be shown in a similar manner that the functional (1.5) is also non-differentiable in general.

Let us turn now to an investigation of the properties of the functions which minimize the functionals (1.3) and (1.5). The existence and uniqueness of such functions will be proved in section 4.

2. Qualitative analysis of the flow in pipes

Flows of a viscous-plastic medium in cylindrical pipes possess a number of peculiarities noted in the investigation of the motion in a circular pipe [14]. Among these, for example, are the presence of a rigid nucleus within the flow domain, and the existence of a critical pressure drop, which must be exceeded in order to maintain stationary motion with a non-zero velocity.

It will be shown below that these peculiarities hold even for the motion of a viscous-plastic medium in cylindrical pipes of arbitrary cross-section, and are associated with the non-differentiability of the functional (1.5).

Let us say that the flow does or does not exist in the domain if

$$\int_\omega |u|\,d\omega > 0, \qquad \int_\omega |u|\,d\omega = 0$$

respectively.

Let us prove the following lemma.

Lemma 2.1. A flow does not exist in the domain ω if and only if

$$\tau_0 \int_\omega |\operatorname{grad} h|\,d\omega - c\int_\omega h\,d\omega \geqslant 0 \tag{2.1}$$

for all h defined in the domain ω and satisfying the boundary condition (1.6).

Proof. The conditions for the existence and absence of a flow in the domain ω are evidently equivalent to the conditions $J(u) < 0$ or $J(u) = 0$ respectively. Let the inequality (2.1) hold. Then $J(h) \geqslant 0$ for all h satisfying condition (1.6). Therefore, $J(u) = 0$. Hence, the sufficiency of condition (2.1) is proved. Now, let us assume that there is no flow in the domain ω and there exists a quantity h satisfying (1.6) such that

$$\tau_0 \int_\omega |\operatorname{grad} h|\,d\omega - c\int_\omega h\,d\omega < 0$$

Then a sufficiently small number $\lambda > 0$ is found for which $J(\lambda h) < 0$. But since the function u minimizes the functional considered, $J(u) \leqslant J(\lambda h)$, that is, the flow exists in ω, which contradicts the assumption made. The lemma is proved.

The inequality (2.1) is essential to an analysis of problems of the motion of a viscous-plastic medium in pipes. It is hence desirable to establish conditions associated with the geometry of the domain ω and the parameters c, τ_0 which would guarantee compliance with the inequality (2.1).

The following lemmas hold.

*Lemma** 2.2.[†] If $h(x, y)$ is a smooth function satisfying condition (1.6), then

$$K \int_\omega |\operatorname{grad} h|\,d\omega \geqslant \int_\omega h\,d\omega, \qquad K = \sup_{\omega' \subseteq \omega} \frac{\operatorname{mes} \omega'}{\operatorname{mes} \Gamma'} \tag{2.2}$$

[†] The proofs of the starred lemmas and theorems are given in the Appendix at the end of the paper.

where ω' is an arbitrary sub-domain† of the domain ω, with boundary Γ'.

*Lemma** 2.3. There exists a sub-domain ω_1 with boundary Γ_1 for which

$$K = \frac{\text{mes } \omega_1}{\text{mes } \Gamma_1}$$

where, if P is a point of Γ_1 not on Γ, then the connected part of the set $\Gamma_1 \setminus \Gamma$, containing P is the arc of a circle touching Γ.

The boundary Γ in lemma 2.3 is assumed to be smooth, except possibly, at a finite number of points.

*Lemma** 2.4. If the domain ω is p-connected, and d is the inner radius of the domain, equal to $\max \rho(P, \Gamma)$, where $\rho(P, \Gamma)$ is the distance between the point P and the domain boundary for all points $P \in \omega$, then the constant K in lemma 2.2 agrees with the estimate $(d/2) \leqslant K \leqslant 8pd$, wherein the lower bound is exact.

It follows from lemmas(2.1) to (2.4) that a necessary and sufficient condition for the absence of flows in the domain ω is the condition $c \leqslant \tau_0 / K$. If $c < \tau_0 / 8pd$, there is no flow in ω. Conversely, if $c > 2\tau_0 / d$, flow always exists in ω.

The equality $c = \tau_0 / K$ defines the critical value of the pressure drop between the ends of the pipe. The necessary and sufficient condition for the existence of a stationary flow with a non-zero discharge is an excess over this critical pressure drop.

Lemma 2.3 permits the value of K to be found effectively for a number of domains. For example, for a circle of radius R, $K = R/2$; for an annulus of inner radius R_2, $K = {}^1/_2 (R_1 - R_2)$; for a square of side a and a rectangle with sides a, b $(a > b)$, respectively

$$K = \frac{a}{2 + \sqrt{\pi}}, \qquad K = \frac{ab}{a + b + \sqrt{(a-b)^2 + \pi ab}}$$

Let us note that out of all pipes with equal cross-sectional areas, the least critical pressure drop is achieved for a circular pipe.

Let us now formulate and prove the following theorem.

Theorem 2.1. If the function $u(x, y)$ minimizes (1.5), then

$$\int_\omega |u| \, d\omega \leqslant \frac{2K}{\mu} (cK - \tau_0) \text{ mes } \omega \qquad (2.3)$$

† The quantities mes ω' and mes Γ' may be considered as the area of the domain ω' and the length of the boundary Γ'. They should more strictly be understood to be the Lebesque measures of the corresponding sets.

Proof. Let us assume that $J(u) < 0$ since otherwise $u \equiv 0$ and inequality (2.3) is obvious. But then

$$\frac{\mu}{2\,\mathrm{mes}\,\omega}\Big(\int_\omega |\operatorname{grad} u|\,d\omega\Big)^2 + \tau_0\int_\omega |\operatorname{grad} u|\,d\omega - cK\int_\omega |\operatorname{grad} u|\,d\omega < 0$$

Therefore

$$\int_\omega |\operatorname{grad} u|\,d\omega \leqslant \frac{2\,\mathrm{mes}\,\omega}{\mu}\,(cK - \tau_0)$$

Using (2.2) and (2.4) we obtain theorem 2.1.

*Theorem** 2.2. The function $u(x, y)$ minimizing (1.5) is continuous in the domain ω.

Up to now only general properties of the minimizing function, including the theorem on its continuity, have been established.

However, from the mechanical viewpoint, the analysis of the structure of the minimizing function is of greatest interest.

*Theorem** 2.3. Any local maximum of the minimizing function is achieved in a domain whose every connected component contains a circle of radius $R_1 = \tau_0 / 8pc$ and does not contain a circle of radius larger than $R_2 = 2\tau_0 / c$, where

$$\max u \leqslant \Big(\frac{8pc}{\tau_0}\Big)^2 \frac{2K\,\mathrm{mes}\,\omega}{\pi\mu}\,(cK - \tau_0)$$

The sub-domain ω' of the domain ω in which $u(x, y)$ reaches the local maximum will be called the nucleus of the flow.

The existence of a nucleus in the flow is an important qualitative characteristic of the motion. It means that during the flow of a viscous-plastic medium through a pipe there will always exist at least one nucleus of finite size for any finite pressure drop, which will move as a solid with constant velocity. The terms containing gradients of $u(x,y)$ in the functional (1.5) will vanish in ω' which indicates that there is no dissipation of mechanical energy within the nucleus.

It is easy to show that each connected component of the nucleus is also simply connected in a simply connected domain ω. Let us mention the sufficient conditions which the domain ω, guaranteeing the existence of a simply connected nucleus, must satisfy. These conditions are based on the important principle of majorizing one flow by another. Let $J_\omega(u)$ denote the functional (1.5), where the subscript indicates the domain in which the functional is defined.

Theorem 2.4. (Majorizing principle). Let u be the function minimizing $J_\omega(u)$, and w the function minimizing $J_\Omega(w)$, where $\omega \subseteq \Omega$. Then $u \leqslant w$ in ω.

Proof. Let us assume that the statement in the theorem is false. Then at some point M in ω we will have $u(M) > w(M)$. Let $H(M)$ denote the connected domain containing point M at which $u > w$.

Evidently $H(M) \subset \omega$. Let us assume that

$$\int_{H(M)} \left(\frac{\mu}{2} |\operatorname{grad} u|^2 + \tau_0 |\operatorname{grad} u| - cu\right) d\omega \geqslant$$
$$\geqslant \int_{H(M)} \left(\frac{\mu}{2} |\operatorname{grad} w|^2 + \tau_0 |\operatorname{grad} w| - cw\right) d\omega$$

Let us consider the function u^* in the domain ω

$$u^* = w \text{ in } H(M), \qquad u^* = u \text{ in } \omega \setminus H(M)$$

It is clear the $J_\omega(u^*) \leqslant J_\omega(u)$. But this inequality contradicts the theorem on the uniqueness of the minimizing function. Therefore

$$\int_{H(M)} \left(\frac{\mu}{2} |\operatorname{grad} u|^2 + \tau_0 |\operatorname{grad} u| - cu\right) d\omega \leqslant$$
$$\leqslant \int_{H(M)} \left(\frac{\mu}{2} |\operatorname{grad} w|^2 + \tau_0 |\operatorname{grad} w| - cw\right) d\omega$$

In this case let us introduce the function w^* in the domain Ω

$$w^* = u \text{ in } H(M), \qquad w^* = w \text{ in } \Omega \setminus H(M)$$

As a result we obtain $J_\Omega(w^*) \leqslant J_\Omega(w)$, which also contradicts the uniqueness theorem. We hence obtain obtain the assertion of theorem 2.4.

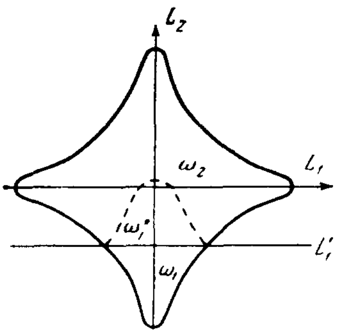

FIG. 2.

The following corollary may be obtained from the majorizing principle. Let K_1 and K_2 be, respectively, the maximum circle contained in ω and the minimum circle containing ω, and let u_1 and u_2 be functions minimizing (1.5) in the circles K_1 and K_2. Then the inequality $u_1 \leqslant u \leqslant u_2$ holds for the function u minimizing the functional (1.5) in the domain ω.

Let us consider the domain ω and some line L dividing this domain into two subdomains ω_1 and ω_2. Let us assume that the domain ω_1^* which is symmetric to the domain ω_1 relative to L, is contained entirely in the domain ω_2 (Fig. 2).

Definition 2.1. If the domain ω is divided into two subdomains ω_2' and ω_2' by any line L' parallel to L and intersecting ω so that $\omega_1^* \subseteq \omega_2'$, then the domain ω is called strongly symmetric in the direction of the line L.

Let $u^*(x, y)$ denote a surface, symmetric to the surface $u(x, y)$ with respect to a vertical plane passing through the line L. The following theorem may be proved in exactly the same way as theorem 2.4.

Theorem 2.5. If $\omega_1^* \subseteq \omega_2$, then we have $u^*(x, y) \subseteq u(x, y)$ in the domain ω_1^*.

Now the condition guaranteeing the existence of a simply-connected flow nucleus may be indicated.

Theorem 2.6. Let the domain ω be strongly symmetric in the l_1 and l_2 directions. Then the minimizing function has a simply connected nucleus.

Proof. It follows from the definition of a domain ω strongly symmetric in the direction l that the ω is simply connected and has the axis of symmetry L parallel to l, where the domain ω_2 is a subdomain containing the point L. Let us prove the theorem by contradiction. Let the flow nucleus contain at least t.vo simply connected components Q_1 and Q_2. Then one of the axes of symmetry, L_1, say, is not an axis of symmetry for one of the domains Q_1 or Q_2. For definiteness, let this be the domain Q_1. Two cases are possible: (1) the axis L_1 intersects Q_1; (2) the axis L_1 does not intersect Q_1. The first case is impossible since, by virtue of theorem 2.5, the subdomains Q_1, having been obtained from the intersection of this domain with the line L_1, should transform into each other under reflection relative to L_1, which denotes the symmetry of Q_1 relative to L_1 and contradicts the assumption made. us show that the second case is also impossible. Indeed, the domain Q_1 is located on one side of the line L_1 and has no boundary strip consisting of points with values less than in Q_1. Let us displace L_1 parallel to itself until it meets the domain Q_1. Let L_1' denote the line which is obtained. It is evident that we arrive at a contradiction to theorem 2.5. when the surface $u = u(x, y)$ is reflected relative to the vertical plane passing through L_1'. The The theorem is proved.

It follows from theorem 2.6 that flows of a viscous-plastic medium in pipes with cross-sections in the form of regular polygons, ellipses, etc. have simply-connected nuclei. A number of domains (not necessarily convex) which are strongly symmetric with respect to directions may be constructed according to the following rule. Let us consider the angl^e^ $r \geqslant 0$, $0 \leqslant \varphi \leqslant \pi / k$ (k is an integer, $k = 2$) in the (x,y) plane, and let us consider a curve in angle such that if (x_1, y_1) and (x_2, y_2) are points on this curve, then

$$\tan\left(\frac{k-2}{2k}\pi\right) \leqslant \frac{y_2 - y_1}{x_2 - x_1}.$$

Let us reflect this curve successively relative to the rays

$$r \geqslant 0, \quad \varphi = \pi s / k \ (s = 1, 2, \ldots, 2k - 1).$$

The closed contour obtained bounds a domain which is strongly symmetric in k directions ($\varphi = 0, \pi / k, \ldots, (k - 1)\pi / k$). The converse statement is also true: Any domain strongly symmetric in k directions may be constructed by the method mentioned. It follows from theorems 2.5 and 2.6 that the flow nucleus in a domain strongly symmetric in k

directions is also strongly symmetric in the same k directions.

In conclusion, let us consider some geometric properties of the flow domain. Let us introduce the following definitions.

Definition 2.2. We call an open connected subdomain ω' of the domain ω in which $u(x, y) > 0$, the connected component of the flow domain.

Let us note that the connected component of the flow does not permit continuous shears and rotations keeping it within ω.

Definition 2.3. A part of the boundary of the connected component of the flow domain is called internal if it has no common points with the boundary of the domain ω.

The following theorem holds.

*Theorem** 2.7. The connected part of the internal boundary of the connected component of a flow domain ω' is convex.

The results obtained enable the following qualitative picture to be given of the motion of a viscous-plastic medium in pipes of arbitrary cross-section.

In the general case, stationary motion of a viscous-plastic fluid is possible in pipes if the pressure drop between the ends of the pipe exceeds some critical value dependent on the yield point of the medium and on the cross-sectional geometry. Within the flow domain there is always at least one nucleus within which the medium is in a rigid state moving as a solid at constant speed. If the pipe cross-section is a simply connected and strongly symmetric domain, such a nucleus will be unique and also simply connected and strongly symmetric. It follows from theorem 2.7 that the flow domain must adjoin the pipe walls, at least at some sections. When the shape of the pipe cross-section has the shape in Fig. 3, the stagnation zone in the extension ABC (i.e., the domain where the flow velocity is zero), if it exists, will have a boundary which is convex towards the extension.

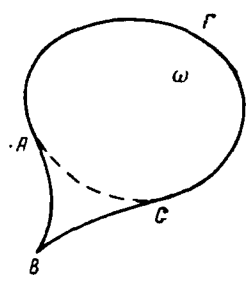

FIG. 3.

Hence, the functional formulation of the problem in the case under consideration turns out to be very effective for the investigation of the qualitative peculiarities of the flow. It should be particularly emphasized that in such an approach the rigid nuclei are obtained as a natural element of the theory, while in the differential-equation formulations of the same problem the existence of such nuclei is not known in advance and requires the introduction into the theory of such elements as the unknown boundaries of these nuclei which are difficult to determine. In constructing a rigorous theory of the flow of a viscous-plastic medium the functional formulation of the problem is, therefore, more complete since it is not associated with the assumptions on the existence or absence of nuclei; the assumption of the existence of such domains is always an additional important physical hypothesis.

3. Plane flow of a viscous-plastic medium around a body

Let us consider the problem of the motion of a body of arbitrary configuration in a plane channel of finite width.

Let $2d$ denote the width of the channel, and v the velocity of the body. In a coordinate system connected rigidly to the moving body, and in the absence of external mass forces the appropriate functional will be

$$J(\Psi) = \int_\omega \left\{ \frac{\mu}{2} [(P\Psi)^2 + (Q\Psi)^2] + \tau_0 [(P\Psi)^2 + (Q\Psi)^2]^{1/2} \right\} d\omega, \qquad (3.1)$$

where the integration extends over the whole strip with the exeption of the domain bounded by the contour Γ (Fig. 4).

On the boundary Γ the function $\Psi(x, y)$ should satisfy the conditions

$$\Psi|_{y=-d} = -Ud, \qquad \Psi|_{y=d} = Ud, \qquad \left.\frac{\partial \Psi}{\partial y}\right|_{y=-d} = \left.\frac{\partial \Psi}{\partial y}\right|_{y=d} = U$$
$$\Psi|_\Gamma = c, \qquad \left.\frac{\partial \Psi}{\partial n}\right|_\Gamma = 0 \qquad (3.2)$$

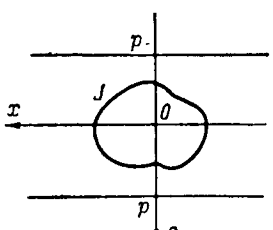

FIG. 4.

Here c is some constant, equal to zero for flow around a symmetric body if its axis of symmetry coincides with the middle line of the channel. We shall seek the solution of the problem in the form

$$\Psi(x, y) = Uy + \varphi(x, y) + u(x, y) \qquad (3.3)$$

The function $u(x, y)$ vanishes together with its normal derivative at $y = \pm d$ and on the contour Γ, and $\phi(x, y)$ is a fixed function, finite in the strip, which satisfies the conditions

$$[\varphi(x, y) + Uy]_\Gamma = c$$
$$\frac{\partial}{\partial n} [\varphi(x, y) + Uy]_\Gamma = 0 \qquad (3.4)$$

on Γ.

Substituting (3.3) into (3.1) we obtain

$$J(u) = \int_\omega \left\{ \frac{\mu}{2} [(P\varphi + Pu)^2 + (Q\varphi + Qu)^2] + \right.$$
$$\left. + \tau_0 [(P\varphi + Pu)^2 + (Q\varphi + Qu)^2]^{1/2} \right\} d\omega \qquad (3.5)$$

Let us analyze some properties of the function $u(x, y)$ which minimizes the functional (3.5).

Let Π_k denote the half-strip $\{x \geqslant k;\ -d \leqslant y \leqslant d\}$, and Π_{sk} the rectangle $\{s \leqslant x \leqslant k,\ -d \leqslant y \leqslant d\}$. Let $v(x, y)$ be continuous and square-symmetric in Π_k together with its partial derivatives to the second order inclusive. Moreover, let

$$\int_{\Pi_k} |Pv|\, d\omega < C, \qquad \int_{\Pi_k} |Qv|\, d\omega < 2c$$

Then the inequalities

$$\begin{aligned}
&\int_{\Pi_k} |Pv|\, d\omega \geqslant \frac{1}{\sqrt{2}\,d} \int_{\Pi_k} \left|\frac{\partial v}{\partial x}\right| d\omega, &&\int_{\Pi_k} |Pv|\, d\omega \geqslant \frac{1}{\sqrt{2}\,d} \int_{\Pi_k} \left|\frac{\partial v}{\partial y}\right| d\omega \\
&\int_{\Pi_k} |Pv|\, d\omega \geqslant \frac{1}{2d^2} \int_{\Pi_k} |v|\, d\omega, &&\int_{\Pi_k} \left|\frac{\partial^2 v}{\partial y^2}\right|^2 d\omega \geqslant \frac{1}{4d^2} \int_{\Pi_k} \left|\frac{\partial v}{\partial y}\right|^2 d\omega \\
&\int_{\Pi_k} (Qv)^2\, d\omega \geqslant \frac{1}{d^2} \int_{\Pi_k} \left|\frac{\partial v}{\partial x}\right|^2 d\omega, &&\int_{\Pi_k} \left|\frac{\partial^2 v}{\partial y^2}\right|^2 d\omega \geqslant \frac{1}{16d^4} \int_{\Pi_k} |v|^2\, d\omega
\end{aligned} \tag{3.6}$$

hold.

Theorem 3.1. The estimate

$$\int_{\Pi_x} \Phi(u)\, d\omega \leqslant \frac{1}{2} J(u)\, C^{\frac{x-L}{p}-1} \quad x > L$$

$$\Phi(u) = \frac{\mu}{2} [(Pu)^2 + (Qu)^2] + \tau_0 [(Pu)^2 + (Qu)^2]^{1/2}$$

where $C < 1$, p is a sufficiently large positive number, and $2L$ is the length of the body in the channel, is valid for the function $u(x, y)$ minimizing (3.5).

Proof. Let us consider the function $h(x)$.

$$h(x) = \begin{cases} 1, \ x \leqslant s, & s \geqslant L \\ -2(k-s)^{-2}(x-s)^2 + 1, & s \leqslant x \leqslant (k+s)/2,\ k > s \\ 2(k-s)^{-2}(x-k)^2, & (k+s)/2 \leqslant x \leqslant k \\ 0, \ x \geqslant k \end{cases}$$

By virtue of the uniqueness theorem on the minimizing function

$$J(u) \leqslant J(uh) \tag{3.7}$$

Using (3.6) and (3.7), taking account of the definition of $h(x)$, we obtain

$$\int\limits_{\Pi_k} \Phi(u)\, d\omega \leqslant$$

$$\leqslant \int\limits_{\Pi_{sk}} \left\{\frac{\mu}{2} C_1\left(\frac{d}{k-s}\right)[(Pu)^2 + (Qu)^2] + \tau_0 C_2\left(\frac{d}{k-s}\right)[(Pu)^2 + (Qu)^2]^{1/2}\right\} d\omega \tag{3.8}$$

$$C_1\left(\frac{d}{k-s}\right) = \frac{4d}{k-s}\left[s + \frac{26d}{k-s} + \frac{32d^2}{(k-s)^2} + \frac{128d^3}{(k-s)^3}\right]$$

$$C_2\left(\frac{d}{k-s}\right) = \left(\frac{36\sqrt{2}\,d}{k-s}\right)^{1/2} + \frac{(4+6\sqrt{2})\,d}{k-s} + \frac{8d^2}{(k-s)^2}$$

Let

$$C\left(\frac{d}{k-s}\right) = \max\left\{C_1\left(\frac{d}{k-s}\right),\ C_2\left(\frac{d}{k-s}\right)\right\}$$

Then the estimate

$$\int\limits_{\Pi_k} \Phi(u)\, d\omega \leqslant C\left(\frac{d}{k-s}\right) \int\limits_{\Pi_{sk}} \Phi(u)\, d\omega \tag{3.9}$$

results from the inequality (3.8).

Now, let us put $k - s = p$. Evidently, $C(d/p) < 1$ for sufficiently large p. Let us fix this number p and let r be a positive integer such that

$$\frac{x-L}{p} - 1 < r \leqslant \frac{x-L}{p}$$

The chain of inequalities

$$\int\limits_{\Pi_x} \Phi(u)\, d\omega \leqslant C\left(\frac{d}{p}\right) \int\limits_{\Pi_{x-p,\,x}} \Phi(u)\, d\omega \leqslant C\left(\frac{d}{p}\right) \int\limits_{\Pi_{x-p}} \Phi(u)\, d\omega \leqslant \dots$$

$$\dots \leqslant \left[C\left(\frac{d}{p}\right)\right]^r \int\limits_{\Pi_{x-rp}} \Phi(u)\, d\omega \tag{3.10}$$

follows from (3.9).

Finally, we obtain the statement of the theorem from (3.10)

$$\int\limits_{\Pi_x} \Phi(u)\, d\omega \leqslant \left[C\left(\frac{d}{p}\right)\right]^{\frac{x-L}{p}-1} \int\limits_{\Pi_L} \Phi(u)\, d\omega \leqslant \frac{1}{2} J(u)\left[C\left(\frac{d}{p}\right)\right]^{\frac{x-L}{p}-1} \tag{3.11}$$

Theorem 3.1 enables an estimate to be made of the rate of decay of the function $u(x, y)$ with distance along the channel from the moving body.

Indeed, let us use the notation

$$Lu = (Pu)^2 + (Qu)^2$$

Let us note that if μ and τ_0 have upper bounds, then $J(u) < k$, where k is independent of μ and τ_0. Hence, it follows that

$$\max_{\Pi_x} |u(x, y)| \leqslant \frac{1}{2} \int_{\Pi_x} |Qu|\, d\omega \leqslant \frac{1}{2} \int_{\Pi_x} (Lu)^{1/2}\, d\omega$$
$$\max_{\Pi_x} |u(x, y)|^2 \leqslant 12d^2 \int_{\Pi_x} Lu\, d\omega \tag{3.12}$$

From inequalities (3.11) and (3.12) we find

$$\frac{\mu}{24d^2} \max_{\Pi_x} |u|^2 + 2\tau_0 \max_{\Pi_x} |u| \leqslant \frac{k}{2}\left[C\left(\frac{d}{p}\right)\right]^{\frac{x-L}{p}-1}$$

The last estimate permits two important qualitative deductions. The first is that the quantity $u(x, y)$ decreases exponentially as $x \to \infty$. This means that the perturbations caused by the body moving in the channel are practically concentrated in some neighborhood of it.

Such a conclusion corresponds qualitatively well with the assumption made in [9] on the localization of perturbations caused by a body moving in a viscous-plastic medium in a domain of finite size, and is confirmed by experimental results [15].

The second qualitative conclusion which may be made on the basis of the last estimate is important in investigations of the passage to the limit as $\tau_0 \to 0$ or $\mu \to 0$.

If we put $\tau_0 = 0$ or $\mu = 0$ in the equations of state of a viscous-plastic medium, these equations will pass over formally into the well-known equations of state of a viscous fluid or an ideally plastic medium. Nevertheless, the assertion that the solution of problems of the flow of a viscous-plastic medium with the corresponding passages to the limit will tend to the solutions of the corresponding problems of the motion of a viscous fluid or an ideally plastic medium is not obvious. The existence of such a tendency would be an additional argument indicating the mathematical and physical correctness of the model of the viscous-plastic medium. Moreover, this circumstance may turn out to be quite useful in working out effective computational methods.

Theorem 3.1 enables us to assert that the function $u(x, y)$ has derivatives to the second order inclusive, which are summable in the domain ω, for all values of the parameter τ_0, including $\tau_0 = 0$. In order to prove this statement it is evidently sufficient to show that $u(x, y)$ has summable partial derivatives to the second order inclusive in Π_L. Let us note that

$$\Pi_x = \bigcup_{k=0}^{\infty} \Pi_{x+kp,\, x+(k+1)p}$$

Then

$$\int_{\Pi_x} (Lu)^{1/2}\, d\omega = \sum_{k=0}^{\infty} \int_{\Pi_{x+kp,\, x+(k+1)p}} (Lu)^{1/2}\, d\omega \leqslant 2pd \sum_{k=0}^{\infty} \Big(\int_{\Pi_{x+kp}} Lu\, d\omega \Big)^{1/2} \leqslant$$
$$\leqslant 2pd \sum_{k=0}^{\infty} \Big\{ \frac{1}{\mu} J(u) \Big[C\Big(\frac{d}{p}\Big) \Big]^{\frac{x-L+kp}{p}-1} \Big\}^{1/2} \tag{3.13}$$

Since the right side in (3.13) is finite, the assertion expressed is actually valid.

Let us now prove that the flow of a viscous-plastic medium transforms as $\tau_0 \to 0$ into the corresponding flow of a viscous fluid both in the case of a channel and in the case of the motion of a viscous-plastic medium in pipes.

Let $J_\tau^{(1)}(u)$ and $J_\tau^{(2)}(u)$, respectively, denote the functionals (1.5) and functional obtained from $J_\tau^{(i)}(u)$ at $\tau_0 = 0$ ($i = 1, 2,$).

Lemma 3.1. The estimates

$$\Big(\frac{\mu}{2}\int_\omega |\operatorname{grad} u_\tau^{(1)}|^2\, d\omega\Big)^{1/2} < k_1, \qquad J_\tau^{(1)}(u_\tau^{(1)}) \geqslant -k_2$$

where k_1 and k_2 are independent of τ_0, are valid for a continuous and square-summable function $u_\tau^{(1)}$, together with its first derivative.

Proof. Since $J_\tau^{(1)}(u)$ is considered in functions given in the bounded domains ω, we have from (1.5)

$$\frac{\mu}{2}\int_\omega |\operatorname{grad} u_\tau^{(1)}|^2 d\omega \leqslant C \int_\omega |u_\tau^{(1)}|\, d\omega \leqslant C \Big(\int_\omega |u_\tau^{(1)}|^2 d\omega\Big)^{1/2} \leqslant$$
$$\leqslant C_2 \Big(\int_\omega |\operatorname{grad} u_\tau^{(1)}|^2\, d\omega\Big)^{1/2}, \qquad J_\tau^{(1)}(u_\tau^{(1)}) \geqslant$$
$$\geqslant \frac{\mu}{2}\int_\omega |\operatorname{grad} u_\tau^{(1)}|^2\, d\omega - C_2 \Big(\int_\omega |\operatorname{grad} u_\tau^{(1)}|^2\, d\omega\Big)^{1/2} \geqslant -\frac{2C_2^2}{\mu}$$

The lemma is proved.

Now let us formulate and prove the following theorem.

Theorem 3.2. The functions $u_\tau^{(i)}$ converge strongly as $\tau_0 \to 0$ to the functions $u_0^{(i)}$ minimizing the functionals $J_0^{(i)}(u)$ in the spaces of functions which are summable together with their partial derivatives to ith order inclusive.

Proof. By virtue of lemma 3.1 and relationships (3.12) and (3.13), we have

$$-c < J_\tau^{(i)}(u_\tau^{(i)}) < J_\theta^{(i)}(u_\theta^{(i)})$$

for $\theta < r$. But then $J_\tau^{(i)}(u_\tau^{(i)}) \to a_i$ as $r_0 \to 0$.

Furthermore, since

$$\left(\frac{\mu}{2}\int_\omega |\operatorname{grad} u|^2 d\omega\right)^{1/2} < C, \qquad \left(\frac{\mu}{2}\int_\omega Lu\, d\omega\right)^{1/2} < C \tag{3.14}$$

then $u_r^{(i)}$ converges weakly as $r_0 \to 0$ to some function $u_*^{(i)}$ in the space of functions square-summable to the ith order inclusive. By virtue of the uniqueness theorem for the minimizing function $J_\tau^{(i)}(u_0^{(i)}) \geqslant J_\tau^{(i)}(u_\tau^{(i)})$. On the basis of known theorems on weak limits [16], we obtain

$$J_0^{(i)}(u_0^{(i)}) \geqslant \lim_{\tau_0 \to 0} J_0^{(i)}(u_\tau^{(i)}) \geqslant J_0^{(i)}(u_*^{(i)})$$

The theorem is proved.

4. Absolute minimum of the functionals

Let us consider the function $u(x)$ defined in the domain ω of the n-space $R^{(n)}$ $[x = (x_1, x_2, \ldots, x_n)]$. Let us assume relative to ω that it has finite width in the variable x_n, i.e., if $x_1 = (x_1^1, x_2^1, \ldots, x_n^1)$ and $x_2 = (x_1^2, x_2^2, \ldots, x_n^2)$ are two points of ω, then $|x_n^1 - x_n^2| < d$. Let ω_N denote the domain consisting of x such that $|x_i| \leqslant N$ $(i \leqslant n-1)$. Let the expressions $\|u\|_1, \ldots, \|u\|_{p+1}$, possessing the properties of the prenorm, i.e., such that

$$\|u + v\|_i \leqslant \|u\|_i + \|v\|_i, \qquad \|\lambda u\|_i = |\lambda|\,\|u\|_i \tag{4.1}$$

be determined by means of functions $u(x)$ given in ω.

Let us consider contraction of the function $u(x)$ in ω_N; let us assume that the expressions (4.1) induce the appropriate prenorms $\|u\|_{1,N}$, $\|u\|_{2,N}, \ldots, \|u\|_{p,N}$, $\|u\|_{p+1,N}$ in ω_N, where

$$\|u\|_{p+1}^2 = \int_\omega u^2 d\omega, \qquad \|u\|_{p+1,N}^2 = \int_{\omega_N} u^2 d\omega \tag{4.2}$$

Let us introduce the Banach spaces $A(\omega)$, $A(\omega_N)$, $B(\omega)$, $B(\omega_N)$, $C(\omega)$ and $C(\omega_N)$, in which the norms are defined as follows:

$$\|u\|_{A(\omega)} = \sum_{i=1}^{p+1} \|u\|_i, \qquad \|u\|_{A(\omega_N)} = \sum_{i=1}^{p+1} \|u\|_{i,N} \tag{4.3}$$

$$\|u\|_{B(\omega)} = \sum_{i=1}^{k} \|u\|_i + \|u\|_{p+1}, \qquad \|u\|_{B(\omega_N)} = \sum_{i=1}^{k} \|u\|_{i,N} + \|u\|_{p+1,N} \tag{4.4}$$

$$\| u \|_{C(\omega)} = \sum_{i=k+1}^{p+1} \| u \|_i, \qquad \| u \|_{C(\omega_N)} \sum_{i=k+1}^{p+1} \| u \|_{i,N} \tag{4.5}$$

The spaces $A(\omega_N)$, $B(\omega_N)$ and $C(\omega_N)$ are the contractions of the spaces $A(\omega)$, $B(\omega)$ and $C(\omega)$ in the domain ω_N.

Let us make the following assumption relative to the introduced prenorms: If $u(x) \in B(\omega)$, then $u(x)$ considered in ω_N, belongs to $B(\omega_N)$, and hence

$$\| u \|_i \geqslant \| u \|_{i,N}, \qquad \lim_{N \to \infty} \| u \|_{i,N} = \| u \|_i \qquad (i = 1, 2, \ldots, k) \tag{4.6}$$

Analogously, if $u(x) \in C(\omega)$, then $u(x) \in C(\omega_N)$ and

$$\| u \|_{i,N} \geqslant \| u \|_{i,N}, \quad \lim_{N \to \infty} \| u \|_{i,N} = \| u \|_i \qquad (i = k+1, \ldots, p+1) \tag{4.7}$$

Let us assume that the spaces introduced possess the following properties.

1°. In the space $B(\omega)$ the sphere $\| u \|_{B(\omega)} \leqslant K$ is weakly compact.

2°. The topology in the space $B(\omega_N)$ is stronger than in the space $C(\omega_N)$, i.e.

$$\| u \|_{B(\omega_N)} \geqslant C_N \| u \|_{C(\omega_N)}$$

3°. The imbedding of the space $B(\omega_N)$ in $L_2(\omega_N)$ is completely continuous.

4°. If $u(x) \in B(\omega)$ and $\| u \|_{p+1} = 0$, then $\| u \|_{B(\omega)} = 0$.

5°. Let some subspace $A_1(\omega)$ be isolated out of $A(\omega)$. The contraction of $A(\omega)$ in ω_N yields $A(\omega_N)$ and that of $A_1(\omega)$ yields $A_1(\omega_N)$. If $v(x)$, considered in ω_N for $x \in \omega$, yields the element $A_1(\omega_N)$ for all N and $\| v \|_{A(\omega_N)} < C$ independently of N, then $v(x) \in A_1(\omega)$.

Let a functional $J(u)$ having the structure

$$J(u) = \Phi(\| u + \varphi_1 \|_1; \ldots, \| u + \varphi_p \|_p) + T(u), \qquad \varphi_i(x) \in A(\omega) \tag{4.8}$$

where the functional $T(u)$ allows expansion in the space $B(\omega)$ and is weakly continous [16], be defined in the space $A(\omega)$. The function $\Phi(\lambda_1, \lambda_2, \ldots, \lambda_p)$ is continous in all the variables and is such that if $0 \leqslant \lambda_i \leqslant a_i$ and $\Phi(\lambda_1, \ldots, \lambda_p) \geqslant \Phi(a_1, \ldots, a_p)$, then $\lambda_i = a_i$. The functional $J(u)$ is assumed to be increasing and bounded from below.

Let us consider the functional $J(u)$ in some subspace $A_1(\omega)$ of the space $A(\omega)$.

Theorem 4.1. The functional $J(u)$ reaches its minimum in $A_1(\omega)$.

Proof. Let us consider the minimizing sequence u_i from $A_1(\omega)$. The functional $J(u)$ is increasing, and therefore

$$\| u_i \|_{A(\omega)} < C \tag{4.9}$$

But then also $\| u_i \|_{B(\omega)} < C$. Since the sphere in $B(\omega)$ is weakly compact, there exists a $u_0 \in B(\omega)$ which is a weak limit of $u_i(x)$

$$u_i \longrightarrow u_0, \qquad i \to \infty, \qquad u_0 \in B\,(\omega) \ \text{ in } B(\omega)$$

Then

$$u_i + \varphi_k \longrightarrow u_0 + \varphi_k, \qquad i \to \infty \text{ in } B(\omega)$$

Let us show that $u_0\,(x) \in A\,(\omega)$. By virtue of the assumptions made relative to the spaces, the function $u_i + \phi_k$ and $u_0 + \phi_k$ may be considered as elements of the spaces $B\,(\omega_N)$ and $A\,(\omega_N)$, where $u_i + \phi_k \to u_0 + \phi_k$, and $i \to \infty$ in $A\,(\omega_N)$. The latter relationships are needed in the proof. Indeed, since $u_i\,(x)$ converges weakly to some limit in $A\,(\omega_N)$, and continuous functionals in $L_2\,(\omega N)$ are continuous functionals in $A\,(\omega_N)$, then $u_0 + \phi_k$ and the weak limit in $A\,(\omega_N)$ yield identical values. Therefore, by virtue of 4°, the relation to be proved is valid. Furthermore, on the basis of known theorems on weak limits [17]

$$\lim \| u_n + \varphi_j \|_{A\,(\omega_N)} \geqslant \| u_i + \varphi_j \|_{A\,(\omega_N)}, \qquad n \to \infty \tag{4.10}$$

From the demands imposed on the function Φ, 5° and (4.9) it follows that

$$\lim \| u_n + \varphi_j \|_{A\,(\omega)} \geqslant \| u_0 + \varphi_j \|_{A\,(\omega)}, \qquad n \to \infty \tag{4.11}$$

Later the following refinement of inequality (4.11) will be required

$$\lim \| u_i + \varphi_j \|_j \geqslant \| u_0 + \varphi_j \|_j \qquad (j = 1, 2, \ldots, p + 1), \qquad i \to \infty$$

Let us consider the Banach spaces $D_i\,(\omega_N)$ and $D_i\,(\omega)$ with the norms

$$\| u \|_{D_i\,(\omega_N)} = \| u \|_{i,\,N} + \| u \|_{p+1,\,N}, \qquad \| u \|_{D_i\,(\omega)} = \| u \|_i + \| u \|_{p+1}$$

Weak convergence in $D_i\,(\omega_N)$ follows from the weak convergence in $B\,(\omega_N)$ since $B\,(\omega_N) \subset D_i\,(\omega_N)$ and $B^*\,(\omega_N) \supset D_i^*\,(\omega_N)$, where the starred values denote the conjugate spaces.

By virtue of assumption 3°, the convergence to $u_0 + \phi_j$ in $L_2\,(\omega_N)$ is strong and from the inequality

$$\lim_{l \to \infty} \| u_i + \varphi_j \|_{D_j\,(\omega_N)} \geqslant \| u_0 + \varphi_j \|_{D_j\,(\omega_N)}$$

follows

$$\lim_{i \to \infty} \| u_i + \varphi_j \|_{j,\,N} \geqslant \| u_0 + \varphi_j \|_{j,\,N}, \qquad \lim_{i \to \infty} \| u_i + \varphi_j \|_j \geqslant \| u_0 + \varphi_j \|_j$$

Let us note that $u_0\,(x) \in A_1\,(\omega)$, since the contraction of u_0 in ω_N belongs to $A_1\,(\omega_N)$ and by virtue of assumption 5° we see that $u_0 \in A_1\,(\omega)$.

Let us prove that a minimum is realized on $u_0\,(x)$. Evidently it may be considered that

$$\lim_{i \to \infty} \| u_i + \varphi_j \|_j = a_j > \| u_0 + \varphi_j \|_j \qquad (j = 1, 2, \ldots, p)$$

Then

$$J(u_0) \geqslant \lim_{i\to\infty} J(u_i) = \Phi(a_1, \ldots, a_p)$$

Furthermore

$$\Phi(\|u_0+\varphi_1\|_1, \ldots, \|u_0+\varphi_p\|_p) \geqslant \Phi(a_1, \ldots, a_p)$$

Therefore, because of the properties of the function Φ, $a_j = \|u_0+\varphi_j\|_j$ and theorem 4.1 is proved. In addition, let us assume that the prenorms $\|\cdot\|_1, \ldots, \|\cdot\|_k$ possess the property of uniform convexity [16], i.e., if $v_1, v_2 \in D_j(\omega)$, then a $\delta(\epsilon)$ corresponds to each $\epsilon > 0$ such that for $\|v_1\|_j = \|v_2\|_j = 1$ and $\|v_1+v_2\|_j \geqslant 2(1-\delta)$ the inequality

$$\|v_1 - v_2\|_j < \varepsilon \tag{4.12}$$

holds.

Moreover, let us assume that we have in the subspace $A_1(\omega)$

$$\|u\|_i \geqslant c\|u\|_{p+1} \qquad (i = 1, 2, \ldots, k)$$

Theorem 4.2. $u_i(x)$ converge strongly in $B(\omega)$.

Proof. Let us consider the elements $(u_i+\varphi_j)/\|u_i+\varphi_j\|_j$, $(u_0+\varphi_j)/\|u_0+\varphi_j\|_j$. Let us note that $\|u_i+\varphi_j\|_j \neq 0$, since if $\|u_0+\varphi_j\|_j = 0$, then $\|u_i+\varphi_j\|_j \to 0$ as $i\to\infty$ and $u_i+\phi_j$ would converge strongly to zero. We have

$$\left\|\frac{u_i+\varphi_j}{\|u_i+\varphi_j\|_j} + \frac{u_0+\varphi_j}{\|u_0+\varphi_j\|_j}\right\|_j \geqslant \frac{\|u_i+2\varphi_j+u_0\|_j}{\|u_i+\varphi_j\|_j} - \frac{|\,\|u_i+\varphi_j\|_j - \|u_0+\varphi_j\|_j\,|}{\|u_i+\varphi_j\|_j}$$

where

$$\lim_{i\to\infty}\frac{\|u_i+2\varphi_j+u_0\|_j}{\|u_i+\varphi_j\|_j} \geqslant 2, \qquad \lim_{i\to\infty}\frac{|\,\|u_i+\varphi_j\|_j - \|u_0+\varphi_j\|_j\,|}{\|u_i+\varphi_j\|_j} = 0$$

Therefore

$$\left\|\frac{u_i+\varphi_j}{\|u_i+\varphi_j\|_j} + \frac{u_0+\varphi_j}{\|u_0+\varphi_j\|_j}\right\|_j \geqslant 2 - \delta(i) \qquad (\delta(i)\to 0 \quad \text{for} \quad i\to\infty)$$

But then, because of the uniform convexity

$$\varepsilon \geqslant \left\|\frac{u_i+\varphi_j}{\|u_i+\varphi_j\|_j} - \frac{u_0+\varphi_j}{\|u_0+\varphi_j\|_j}\right\|_j \geqslant \frac{\|u_i-u_0\|_j}{\|u_i+\varphi_j\|_j} - \left|\frac{\|u_i+\varphi_j\|_j - \|u_0+\varphi_j\|_j}{\|u_i+\varphi_j\|_j}\right| \tag{4.13}$$

which proves the theorem of the strong convergence of u_i in $B(\omega)$.

Theorem 4.3. If the prenorm has the form $\|u\|^2 = (u, u)$, where (u, v) is a bilinear positive symmetric form, then this prenorm is uniformly convex.

Proof. Let

$$\|u\| = \|v\| = 1, \quad \|u+v\| > 2(1-\delta), \quad \text{for} \quad (u, v) \geqslant 1 - 4\delta + 2\delta^2$$

Then

$$\| u - v \| = \sqrt{2 - 2(u, v)} < 2\sqrt{2\delta}$$

Let us mention some additional restrictions on the prenorm $\|\cdot\|_j$, $j \geqslant k + 1$, which will guarantee the strong convergence of u_i to u_0 in $A(\omega)$.

(*a*) Let the prenorms $\|\cdot\|_j$, $j > k$ be determined by functions $u(x)$, where $x \in \omega \setminus \omega_N$. Let $\|\cdot\|_{j, N^\circ}$ denote these prenorms.

(*b*) $\lim \| u \|_{j, N^\circ} = 0$, $u \in D_j(\omega)$, $N \to \infty$.

(*c*) A $\delta(\| u \|_j, \varepsilon)$ corresponds to each $\epsilon > 0$ such that if $|\| u \|_j - \|u\|_{j, N}| < \delta$, then $\| u \| \varepsilon_{j, N^\circ} < \varepsilon$, where $\delta(\| u \|_j, \varepsilon)$ depends continuously on $\| u \|_j$ for $\| u \|_j \neq 0$.

(*d*) $\| u \|_j \leqslant f[\varphi(\| u \|_{j, N}) + \Psi(\| u \|_{j, N^\circ})]$ and $\varphi(0) = \Psi(0) = f(0) = 0$; the functions f, ϕ, Ψ are continuous.

(*e*) If $\lim \| u_i \|_j = 0$, then $\lim \| u_i \|_{j, N^\circ} = 0$ as $i \to \infty$

Theorem 4.4. Under the assumptions made relative to the prenorms $\|\cdot\|_j$ the sequence $u_i(x)$ converges to $u_0(x)$ strongly in $D_j(\omega)$.

Proof. It follows from theorem 4.3 that there is strong convergence in $L_2(\omega)$.

It is therefore necessary to prove that

$$\lim \| u_i - u_0 \|_j = 0 \qquad \text{for} \quad i \to \infty \tag{4.14}$$

Furthermore, we have from (*d*)

$$\| u_i - u_0 \|_j \leqslant f[\varphi(\| u_i - u_0 \|_{j, N}) + \Psi(\| u_i - u_0 \|_{j, N^\circ})]$$
$$\| u_i - u_0 \|_{j, N^\circ} \leqslant \| u_i + \varphi_j \|_{j, N^\circ} + \| u_i + \varphi_j \|_{j, N^\circ}$$

If $\| u_0 + \varphi_j \|_j = 0$, then $\lim \| u_i + \varphi_j \|_{j, N^\circ} = 0$ as $i \to \infty$, and since $\| u_i - u_0 \|_{j, N} \to 0$, as $i \to \infty$, it follows from the preceding inequalities that $\| u_i - u_0 \|_{j, N} \to 0$ as $i \to \infty$. Let $\| u_0 + \varphi_j \|_j \neq 0$. Then for sufficiently large N

$$\| u_0 + \varphi_j \|_{j, N^\circ} \leqslant \delta(N)$$

Moreover, let N be so very large that

$$|\| u_0 + \varphi_j \|_j - \| u_0 + \varphi_i \|_{j, N}| < {}^1\!/_3\delta$$

Upon selecting sufficiently large values of i we will have

$$|\| u_i + \varphi_j \|_j - \| u_0 + \varphi_j \|_j| < {}^1\!/_3\delta, \qquad |\| u_i + \varphi_j \|_{j, N} - \| u_0 + \varphi_j \|_{j, N}| < {}^1\!/_3\delta$$

Therefore

$$|\| u_i + \varphi_j \|_{j,N} - \| u_i + \varphi_j \|_j | < \delta$$

From condition (*c*) it follows that $\| u_i + \varphi_j \|_{j,N^\circ} < \varepsilon$.

Hence, the arguments of the functions ϕ and ψ, and therefore, of f as well, become as small as desired for sufficiently large values of i. The theorem is proved.

Definition. The functional $J(u)$ is called strongly convex if

$$J[{}^1/_2 (u_1 + u_2)] < {}^1/_2 [J(u_1) + J(u_2)] \quad \text{for} \quad u_1 \neq u_2$$

Uniqueness theorem. If the functional has the property of strong convexity and $a = J(u_1) = J(u_2) = \inf J(u)$, where $u, u_1, u_2 \in A_1(\omega)$, then $u_1 = u_2$.

Proof. Since u_1 and u_2 belong to $A_1(\omega)$, then ${}^1/_2 (u_1 + u_2) \in A_1(\omega)$.

But then

$$J[{}^1/_2 (u_1 + u_2)] < {}^1/_2 [J(u_1) + J(u_2)] = a$$

which contradicts the condition of the theorem.

Let us turn now to the application of the results obtained to the investigation of specific flows of a viscous-plastic medium. Let us use the following notation in (3.5).

$$\| u + \varphi \|_1 = \left\{ \int_\omega \frac{\mu}{2} [(P(\varphi + u))^2 + (Q(\varphi + u))^2] \, d\omega \right\}^{1/2}$$

$$\| u + \varphi \|_2 = \tau_0 \int_\omega [(P(\varphi + u))^2 + (Q(\varphi + u))^2]^{1/2} d\omega$$

Then

$$J(u) = \| u + \varphi \|_1^2 + \| u + \varphi \|_2 \tag{4.15}$$

Hence, $J(u)$ is a particular case of the functional (4.8), where $\Phi(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2$ and, evidently satisfies the demands imposed on this function.

The functional $T(u)$ is zero in the case under consideration.

Let us verify conditions 1° to 5°. We have $p = 2$, $k = 1$. The space $B(\omega)$ is a Hilbert space, and therefore, 1° is satisfied. Condition 2° follows trivially from the Cauchy-Buniakovskii inequality, where $C_N = \text{mes}\, \omega_N$. Conditions 3° and 4° are valid because of well-known imbedding theorems [18].

Let us consider the closure of the space $C_0^\infty(\omega)$ in the metric $A(\omega)$ as the subspace $A_1(\omega)$. The validity of condition 5° is hence obvious.

A stronger assertion is valid relative to the prenorm $\|\cdot\|_1$ namely, it is uniformly convex by virtue of theorem 4.3.

Let us show now that conditions (*a*) to (*e*) are satisfied. Let us define $\| u \|_{j,\ N^\circ}$ thus:

$$\| u \|_{1,\ N^\circ} = \left\{ \int\limits_{\omega \setminus \omega_N} \frac{\mu}{2} [(Pu)^2 + (Qu)^2]\, d\omega \right\}^{1/2}, \qquad \| u \|_{2,\ N^\circ} = \tau_0 \int\limits_{\omega \setminus \omega_N} [(Pu)^2 + (Qu)^2]^{1/2} d\omega$$

Conditions (*a*) and (*b*) are hence evidently satisfied. Since $\|u\|_i \geqslant \|u\|_{i,\ N^\circ}$, here, condition (*e*) is also satisfied. Conditions (*c*) and (*d*) follow directly from the following obvious relationships

$$\| u \|_1 - \| u \|_{1,\ N} \geqslant \frac{\| u \|_{1,\ N}^2}{2 \| u \|_1}, \quad \| u \|_2 - \| u \|_{2,\ N} = \| u \|_{2,\ N^\circ}, \quad \| u \|_1 = (\| u \|_{1,\ N}^2 + \| u \|_{1,\ N^\circ}^2)^{1/2}$$

Let us verify the last condition of stric convexity of the functional. Actually

$$\begin{aligned} \| {}^1/_2\, (u_1 + u_2) \|_2 &\leqslant {}^1/_2\, (\| u_1 \|_2 + \| u_2 \|_2) \\ \| {}^1/_2\, (u_1 + u_2) \|_1^2 &\leqslant {}^1/_2\, (\| u_1 \|_1^2 + \| u_2 \|_1^2) \end{aligned}$$

where the equality in the last relationships holds only for $u_1 = u_2$.

Hence, the existence and uniqueness of the solution of the problem of motion of a body in a plane channel representable in the form (2.7) have been proved, where the sum of the last two components is a function from the space $W_2^{(2)}(\omega) \cap W_1^{(2)}(\omega)$. On the basis of known imbedding theorems, the solution is a continuous function.

Let us represent the functional (1.5) as

$$\begin{aligned} J(u) &= \| u \|_1^2 + \| u \|_2 + T(u), \quad \| u \|_1^2 = \int\limits_\omega \frac{\mu}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] d\omega \\ \| u \|_2 &= \tau_0 \int\limits_\omega \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{1/2} d\omega, \qquad T(u) = - \int\limits_\omega cu\, d\omega \end{aligned} \tag{4.16}$$

for the case of flow in a pipe.

Evidently, the functional $T(u)$ is weakly continuous. It follows from (4.16) that $J(u)$ is a particular case of the more general functionals considered above, and it satisfies all the necessary conditions. In exactly the same way as in the case of flow in a channel, we may show that in the case under consideration uniqueness and existence theorems hold for the solution. The sole difference from the preceding case is the fact that the solution will be a function of the space $\overset{0}{W}{}_2^{(1)}(\omega)$.

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APPENDIX

Proof of lemma 2.2. Let us show that $K \geqslant \sup (\text{mes}\, \omega' / \text{mes}\, \Gamma')$, $\omega' \subseteq \omega$. To do this, let us consider an arbitrary domain ω_1 in the domain ω, and the sequence of functions $v_j(x, y)$ which equal 1 in the domain $\omega_1 \setminus O_j(\Gamma_1)$ and 0 in the domain $\omega \setminus (\omega_1 \cup O_j(\Gamma_1))$, where Γ_1 is the boundary of the domain ω_1; $O_j(\Gamma_1)$ is the circle Γ_1 which shrings to Γ_1 as $j \to \infty$. Let us assume the existence of a finite curvature at all points of Γ_1. This permits the introduction of a curvilinear coordinate system in the circle $O_j(\Gamma_1)$ by considering s, the arclength along Γ_1, as one of the coordinates, and n, the length of a segment normal to Γ_1, as the other. The functions $v_j(x, y)$ are monotone functions of the variable n in the circle $O_j(\Gamma_1)$ (the lines $n = \pm\alpha_j$ are the boundary of $O_j(\Gamma_1)$.

Then

$$\lim_{j\to\infty} \int_\omega |\operatorname{grad} v_j|\, d\omega = \lim_{j\to\infty} \int_{O_j(\Gamma_1)} |\operatorname{grad} v_j|\, d\omega = \lim_{j\to\infty} \int_{O_j(\Gamma_1)} \frac{dv_j}{dn} J(s, n)\, ds dn$$

where $J(s, n) = 1 + O(n)$. Hence

$$\lim_{j\to\infty} \int_\omega |\operatorname{grad} v_j|\, d\omega = \lim_{j\to\infty} \int_{O_j(\Gamma_1)} \frac{dv_j}{dn}\, ds dn = \operatorname{mes} \Gamma_1 \tag{A.1}$$

Furthermore, it is evident that

$$\lim_{j\to\infty} \int_\omega v_j dx dy = \operatorname{mes} \omega_1 \tag{A.2}$$

It follows from (A.1) and (A.2) that $K \geqslant \operatorname{mes} \omega_1 / \operatorname{mes} \Gamma_1$. Let us show that $K \leqslant \sup$ $(\operatorname{mes} \omega' / \operatorname{mes} \Gamma')$, $\omega' \subseteq \omega$. It is easy to see that it is sufficient to verify the assertion of lemma 3.1 only by functions h, continuous, positive and belonging to the space $\overset{\circ}{W}{}_p^{(1)}$ $(p > 2)$. Let us continue the function h to zero in some square containing the domain ω. Let us approximate this continued function by polynomials $P_n^1(x, y)$ in the square in the metric of the space $W_p^{(1)}$. Since $p > 2$, it follows from the imbedding theorem that the convergence of $P_n^1(x, y)$ to $h(x, y)$ is uniform. Hence, it may be considered that $|P_n^1(x, y) - h(x, y)| < 1/n$ in ω and

$$\int_\omega |\operatorname{grad}(P_n^1(x, y) - h(x, y))|\, d\omega \leq 1/n$$

Let us put $P_n(x, y) = P_n^1(x, y) - 1/n$. Evidently $P_n(x, y) < 0$ on the boundary Γ of the domain ω. Let us consider the line of zeros of the polynomial $P_n(x, y)$ in the domain ω and bounding the domain s_n containing all the rest of the zeros of $P_n(x, y)$ (this line may consist of a finite number of connected components). Let Q_n denote a function defined in ω as follows:

$$Q_n = \begin{cases} P_n & \text{в } s_n \\ 0 & \text{в } \omega \setminus s_n \end{cases}$$

It is easy to see that $|Q_n - h| < 2/n$ in ω. In fact, we have $|Q_n - h| = |P_n - h| < 2/n$ in s_n. We have $|Q_n - h| = |h|$ in the domain $\omega \setminus s_n$ and since $P_n \leqslant 0$ in $\omega \setminus s_n$, and $h \geqslant 0$, then $|h| < 2/n$. Indeed, let us show that $\operatorname{mes}\{(\omega \setminus s_n) \cap \operatorname{supp} h\} \to 0$ as $n \to \infty$, where supp h is the domain in which $h \geqslant 0$. In fact, let us consider the set $A_n = \{(x, y) : |h| < 2/n\} \cap \text{supp}\, h$. Evidently $(\omega \setminus s_n) \cap \text{supp}\, h \subseteq A_n$ and $A_n \supseteq A_{n+1}$. Since $\lim_{n\to\infty} A_n$ is the empty set and A_n are bounded, then

$$\lim_{n\to\infty} \text{mes}\, A_n = 0$$

It hence follows that

$$\lim_{n\to\infty} \text{mes}\, \{(\omega \setminus s_n) \cap \text{supp}\, h\} = 0 \tag{A.3}$$

Let us show that

$$\lim_{n\to\infty} \int_\omega |\text{grad}\,(h - Q_n)|\, d\omega = 0 \tag{A.4}$$

In fact

$$\int_\omega |\text{grad}\,(h - Q_n)|\, d\omega = \int_{s_n} |\text{grad}\,(h - P_n)|\, d\omega + \int_{(\omega \setminus s_n) \cap \text{supp}\, h} |\text{grad}\, h|\, d\omega \tag{A.5}$$

The first term on the right in (A.5) evidently tends to zero. The second term tends to zero because of (A.3). The next step in the proof is to obtain the inequality

$$K \int_\omega |\text{grad}\; Q_n|\, d\omega \geqslant \int_\omega Q_n d\omega \tag{A.6}$$

Let us introduce the function Q_n^* as follows: Q_n^* is obtained from Q_n by shrinking the local minima by horizontal planes, i.e., if Q_n has a local minimum, then the level lines of Q_n are ovals in the neighborhood of this minimum. Let us select those ovals on which Q_n takes its maximum value.

We put the function Q_n^* equal to this greatest value in the whole domain bounded by the mentioned oval. We repeat this construction for all local minima. Since

$$\int_\omega |\text{grad}\; Q_n|\, d\omega \geqslant \int_\omega |\text{grad}\; Q_n^*|\, d\omega, \qquad \int_\omega Q_n^* d\omega \geqslant \int_\omega Q_n d\omega$$

the inequality (A.6) will follow from the inequality

$$K \int_\omega |\text{grad}\; Q_n^*|\, d\omega \geqslant \int_\omega Q_n^* d\omega \tag{A.7}$$

It is easy to see that the whole domain ω is covered by level lines of the function Q_n^* among which are only a finite number which do not have finite curvature in at least one point. We shall designate such lines as singular. The remaining level lines, which have finite curvature at each point, will be called nonsingular. Among the singular level lines are individual points, simple arc, re-entrant curves. Let us consider the nonsingular level line L_p enclosing the domain ω_p and consisting generally of a finite number of ovals. A curvilinear coordinate system may be introduced in some neighborhood of L_p by considering s, the arclength of the curve L_p, as one coordinate, and n, segments normal to L_p ($n>0$ in the domain ω_p, $n<0$ otherwise), as the other. Evidently, $dxdy = Jdsdn$ here and $J(s, n) \to 1$ as $n \to 0$. In the neighborhood mentioned let us consider another level line $L_{p+\Delta p}$, whose equation $n = n(s) > 0$. Let $Q_n^*(L_p)$ denote the value of Q_n^* on L_p. Let us denote the domain between L_p and $L_{p+\Delta p}$ by $\omega_{p,p+\Delta p}$. Then

$$\int\limits_{\omega_{p,p+\Delta p}} |\operatorname{grad} Q_n^*|\, d\omega \geqslant \int\limits_{\omega_{p,p+\Delta p}} \left|\frac{\partial Q_n^*}{\partial n}\right| Jdsdn = \tag{A.8}$$
$$= [Q_n^*(L_{p+\Delta p}) - Q_n^*(L_p)] \text{ mes } L_p + O(n(s)) \text{ mes } \omega_{p,p+\Delta p}$$

Let us rewrite (A.8) as

$$\frac{\text{mes } \omega_p}{\text{mes } L_p} \int\limits_{\omega_{p,p+\Delta p}} |\operatorname{grad} Q_n^*|\, d\omega \geqslant [Q_n^*(L_{p+\Delta p}) - Q_n^*(L_p)] \text{ mes } \omega_p + \tag{A.9}$$
$$+ O(n(s)) \text{ mes } \omega_{p,p+\Delta p} \text{ mes } \omega_p / \text{mes } L_p$$

Since $\text{mes } \omega_p / \text{mes } L_p < K$, then

$$K \int\limits_{\omega_{p,p+\Delta p}} |\operatorname{grad} Q_n^*|\, d\omega \geqslant [Q_n^*(L_{p+\Delta p}) - Q_n^*(L_p)] \text{ mes } \omega_p + \tag{A.10}$$
$$+ O(n(s)) \text{ mes } \omega_{p,p+\Delta p} \text{ mes } \omega_p / \text{mes } L_p$$

To small-measure accuracy, the domain ω may be partitioned by nonreentrant subdomains $\omega_{p,p+\Delta p}$, i.e. $\omega_\varepsilon = \bigcup \omega_{p,p+\Delta p}$ and $\text{mes}(\omega \diagdown \omega_\varepsilon) < \varepsilon$. Summing (A.10) over different $\omega_{p,p+\Delta p}$ and selecting $n(s)$ sufficiently small, we obtain

$$K \int\limits_\omega |\operatorname{grad} Q_n^*|\, d\omega \geqslant \int\limits_\omega Q_n^* d\omega - \delta \tag{A.11}$$

where $\delta = \delta(\varepsilon, n(s))$, and since the rest of the expressions in (A.11) are independent of ϵ, $n(s)$, then (A.6) follows from (A.11). Since Q_n approximates L in a corresponding manner, the assertion of lemma 3.1 is proved.

Proof of lemma 2.3. Let us note that there exists a subdomain ω_1 of the domain ω in which a value K is achieved (see lemma 2.2). This results from the compactness of the curves with bounded length. Let the point M belong to Γ_1 and not to Γ. Then M is submerged together with its neighborhood $O(M)$ in ω. It is easy to see that part of Γ_1, falling in $O(M)$, should be convex. Furthermore, let M_1 and M_2 be two points which lie sufficiently closely on Γ_1 and belong to $O(M)$. By virtue of the extremal properties of a circle, the arc

M_1M_2 of the boundary Γ_1 is part of a circle. Let us show that Γ_1 and Γ are tangent at common points. Let us assume the opposite (Fig. 5). Let $\alpha \neq 0$, $\gamma = OM_1 + OM_2$; l is the length of the segment M_1M_2; the remaining notation is given in Fig. 5. Then

$$\frac{\text{mes}\,\omega_1}{\text{mes}\,\Gamma_1} \geqslant \frac{\text{mes}\,\omega_1 - \text{mes}\,\omega_1'}{\text{mes}\,\Gamma_1 - \text{mes}\,\gamma + l} = \frac{\text{mes}\,\omega_1 - O(a^2)}{\text{mes}\,\Gamma_1 - a\,[2 - \sqrt{2 + 2\cos\alpha}] + O(a^2)}$$

or

$$a\,[2 - \sqrt{2 + 2\cos\alpha}] \leqslant O(a^2)$$

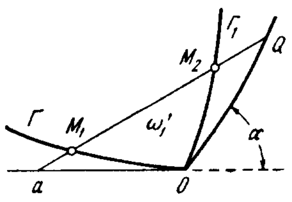

FIG. 5.

The latter inequality is impossible for small values of a. The lemma 2.3 is proved.

Proof of lemma 2.4. The lower bound of K is evident since if d is the inner radius of the domain ω, a circle of radius d may be taken as the subdomain ω'. Let us prove the validity of the upper bound of K. Let us cover a plane by squares with side $4d$. It may happen that the whole domain falls into one square. We will then construct the domain ω on one of the sides of this square. Let the magnitude of the projection be t.

Evidently mes $\omega \leqslant 4td$ and mes $\Gamma \geqslant 2t$, from which mes ω/mes $\Gamma \leqslant 2d$. The assertion of the lemma is satisfied in this case. Let the domain ω be covered by at least two squares. Let us consider squares all of whose sides lie within ω. It is easy to see that the partition of the plane into squares may always be chosen so that the number of such squares will not exceed $p - 1$, where p is the number of connectivities of the domain ω. Let us eliminate these squares from the considerations. Let us take an arbitrary square from the rest, which has at least one common point with the domain ω. Two cases may be represented: (a) the center of the square belongs to ω, (b) the center of the square does not belong to ω. Let us note that the partition of the plane into squares may be selected in such a manner that at least one square of the first kind will take part in covering ω. Let us consider case (a). It is clear that the area of the part of the domain ω included within the square will not exceed $16d^2$, and the length of the part of the boundary Γ falling in the square will not be less than $2d$. Let us consider case (b). Let us construct a concentric square with side a in the selected square. Let us enlarge a from zero until it just reaches the domain ω. Then the area of the part of the domain ω included in the original square will not exceed $16d^2 - a^2$, and the length of the part of the boundary Γ falling into this square will not be less than $4d - a$.

Let the domain ω be covered by r_1 squares of the first kind ($r_1 \geqslant 1$) and by r_2 squares of the second kind. Then

$$\frac{\text{mes}\,\omega}{\text{mes}\,\Gamma} \leqslant \frac{16r_1d^2 + (16d^2 - a^2)\,r_2 + (p-1)\,16d^2}{2r_1d + (4d - a)\,r_2} \leqslant 8pd$$

Lemma 2.4 is proved.

Proof of theorem 2.2. Let us consider the sequence of functions $u_n(x, y)$ from $C_0^{\infty}(\omega)$, which approximates the function $u(x, y)$ in the metric of the space $W_2^{(1)}(\omega)$. Let us transform from the function $u_n(x, y)$ to $u_n^*(x, y)$, obtained from $u_n(x, y)$ by joining local minima to horizontal planes (as was done in the proof of lemma 2.2. in passing from Q_n to Q_n^*). Evidently $J(u_n^*) = J(u_n)$, where $J(u)$ is the functional (1.5). Let us transform from the function $u_n^*(x, y)$ to $v_n(x, y)$ obtained from $u_n^*(x, y)$ by sectioning the local maxima by horizontal planes, where the sectioning of the local maximum may be continued until the inner radius of the section d does not equal $\tau_0/8pc$. The first step in the proof of theorem 2.2. is that

$$J(v_n) \leqslant J(u_n^*) \tag{A.12}$$

The problem formulated is local, i.e., it is necessary to show that the values of the functional (1.5) decrease in the successive sectioning of one local maximum of the function u_n^* if the inner radius d of the sections does not exceed $\tau_0/8pc$. Let $h(x, y)$ be an arbitrary continuous function belonging to the space $W_2^{(1)}(\omega)$ and let $g(x, y)$ be obtained from $h(x, y)$ by sectioning the local maximum of $h(x, y)$ at the heigh H. Let A be the domain of the cut, i.e., the domain in which the functions $h(x, y)$ and $g(x, y)$ are different. If the inner radius d of the domain A does not exceed $\tau_0/8pc$, then

$$J(g) \leqslant J(h) \tag{A.13}$$

Evidently the inequality (A.13) follows from the inequality

$$\int_A \left\{\frac{\mu}{2} |\operatorname{grad} h|^2 + \tau_0 |\operatorname{grad} h| - ch\right\} d\omega \geqslant -\int_A H d\omega \tag{A.14}$$

Let $\psi(x, y)$ denote $h - H$. Then (A.14) is rewritten as

$$\int_A \left\{\frac{\mu}{2} |\operatorname{grad}\psi|^2 + \tau_0 |\operatorname{grad}\psi| - c\psi\right\} d\omega \geqslant 0 \tag{A.15}$$

Let us strengthen the inequality (A.15)

$$\frac{\tau_0}{c}\int_A |\operatorname{grad}\psi|\, d\omega \geqslant \int_A \psi d\omega \tag{A.16}$$

By virtue of lemmas 2.2, 2.4, the inequality (A.16) is satisfied if $\tau_0/c > 8pd$ or $d < \tau_0/8pc$, where d is the inner radius of the domain A. Hence, by virtue of inequality (A.12), the sequence $v_n(x, y)$ is minimizing for (1.5) and, therefore, according to theorem 4.4 the sequence $v_n(x, y)$ converges to $u(x, y)$ in the metric of the space $W_2^{(1)}(\omega)$. Let us proceed now to the proof of the continuity of the function $u(x, y)$. Let us consider an arbitrary point (x_0, y_0) in ω and let us construct two circles K_ρ and $K_{2\rho}$ with radii ρ and 2ρ, respectively, with this point as center. Let us select the quantity ρ so small that $K_{2\rho}$ would lie in ω and $\rho < \tau_0/16pc$. With such a choice of ρ a level line of the function $v_n(x, y)$

intersecting the boundary of the circle $K_{2\rho}$ will pass through each point of the circle K_ρ.

Actually, if a point existed in the circle K_ρ from which a level line did not emergy to intersect the circle of radius 2ρ, then local minima or maxima would exist in the circle $K_{2\rho}$, which may not be by virtue of the construction of the function $v_n(x, y)$. Let

$$\inf v_n(x, y) = v_n(x_1, y_1), \qquad \sup v_n(x, y) = v_n(x_2, y_2), \qquad (x,y)\in K_\rho$$

Let K_r denote a circle of radius r with center at the point (x_0, y_0) and $\rho \leqslant r \leqslant 2\rho$. Let level lines issue from the points (x_1, y_1), (x_2, y_2). These lines will intersect the boundary of K_r at the points $(r, \varphi_1(r))$, $(r, \varphi_2(r))$. Evidently

$$\operatorname{osc} v_n(x, y) = \int_{\varphi_1(r)}^{\varphi_2(r)} \frac{\partial v_n}{\partial \varphi}\, d\varphi, \qquad (x,y)\in K_\rho$$

where $\operatorname{osc} v_n(x, y)$, $(x, y) \in K_\rho$ is the oscillation of the function v_n in the circle K_ρ.

Furthermore

$$[\operatorname{osc} v_n(x, y)]^2 \leqslant 2\pi \int_0^{2\pi} r^2\, |\operatorname{grad} v_n|^2\, d\varphi, \qquad (x,y)\in K_\rho$$

or

$$[\operatorname{osc} v_n(x, y)]^2 \ln 2 = \int_\rho^{2\rho} [\operatorname{osc} v_n(x, y)]^2 \frac{dr}{r} \leqslant 2\pi \int_{K_{2\rho} / K_\rho} |\operatorname{grad} v_n|^2\, d\omega, \qquad (x,y)\in K_\rho$$

Since

$$\lim_{n\to\infty} \int_\omega |\operatorname{grad} v_n|^2\, d\omega = \int_\omega |\operatorname{grad} u|^2\, d\omega$$

then by virtue of the absolute continuity of the integral as a function of the set, N and δ may be associated with any $\epsilon > 0$ such that $\rho < \delta$, $n > N$ and

$$[\operatorname{osc} v_n(x, y)]^2 \ln 2 \leqslant \varepsilon, \quad (x,y)\in K \tag{A.17}$$

By virtue of condition (A.17) and the boundary condition (1.6) to which all the functions v_n are subject, the sequence $v_n(x, y)$ is uniformly bounded and equicontinuous. Using the Artsel theorem, we deduce that the function $u(x, y)$ is continuous.

Proof of theorem 2.3. Let us consider the sequence $v_n(x, y)$ constructed in the proof of theorem 2.2. Since the function $v_n(x, y)$ is continuous in ω, non-negative and vanishes on the boundary Γ, the absolute maximum of $v_n(x, y)$ is achieved in some set each of whose connected components contains a circle of radius $r_0/8pc$. It is easy to see that the function $u(x, y)$, being the uniform limit of the sequence $v_n(x, y)$, possesses the same property. A stronger statement can evidently be made relative to the function $u(x, y)$.

Namely, each local maximum of the function $u(x, y)$ is achieved in a set, each of whose connected components contains a circle of radius $\tau_0/8pc$.

Theorem 2.1 states that

$$\int_\omega |u|\, d\omega \leqslant \frac{2K}{\mu} \operatorname{mes} \omega\, (cK - \tau_0)$$

But since

$$\int_\omega |u|\, d\omega \geqslant \max |u| \left(\frac{\tau_0}{8pc}\right)^2 \pi$$

then

$$\max |u| \leqslant \left(\frac{8pc}{\tau_0}\right)^2 \frac{2K \operatorname{mes} \omega\, (cK - \tau_0)}{\mu\pi}$$

Let us show that the set A on which the local maximum of $u(x, y)$ is achieved may not contain a circle of radius greater than $2\tau_0/c$. Let us assume the opposite. Let A contain a circle K_r of radius $r = (2\tau_0/c) + \epsilon$.

When the domain ω is a circle, the explicit form of the function is known, which minimizes the functional (1.5) [14], where this function is not zero if $r > 2\tau_0/c$. Let us consider the function $u_0(x, y)$, equal to the known minimizing function in the circle K_r and zero outside this circle. It is then easy to verify that $J(u_0 + u) < J(u)$. Thus, in reality, the set A may not contain a circle of radius greater than $2\tau_0/c$. Theorem 2.3 is proved.

Before proceeding to the proof of theorem 2.7, let us prove the following lemma.

Lemma A.1. Let two functions $f(x)$ and $g(x)$ be defined in $[a, b]$ such that

$$f(a) = g(a), \quad f(b) = g(b), \quad f(x) \geqslant g(x), \quad f''(x) \geqslant 0$$

Then

$$\int_a^b |f'|^\alpha\, dx \leqslant \int_a^b |g'|^\alpha\, dx, \qquad \alpha \geqslant 1 \tag{A.18}$$

Proof. Let us prove (A.18), when $f(x) = kx + b$. It follows from the Hölder inequality

$$\int_a^b |g'|^\alpha\, dx \geqslant \left(\frac{1}{b-a}\right)^{\alpha/\alpha'} \left(\int_a^b |g'|\, dx\right)^\alpha, \qquad 1/\alpha + 1/\alpha' = 1$$

But

$$\left(\frac{1}{b-a}\right)^{\alpha/\alpha'} \left(\int_a^b |g'|\, dx\right)^\alpha \geqslant \left(\frac{1}{b-a}\right)^{\alpha/\alpha'} (|g(b) - g(a)|) = k^\alpha (b - a)$$

On the other hand

$$\int_a^b | f' |^\alpha \, dx = k^\alpha (b - a)$$

The proof for $f(x) = kx + b$ has thereby been concluded. Let $a < x_0 < b$. Let us draw a tangent line to $f(x)$ at the point x_0. This tangent lies under the curve $f(x)$ (since $f''(x) \geqslant 0$) and intersects the curve $g(x)$ in at least two points x_0^1, x_0^2. Let us transform from the curve $g(x)$ to the curve $g_1(x)$, which coincides with $g(x)$ outside the segment $[x_0^1, x_0^2]$ and coincides with the tangent to $f(x)$ at the point x_0. Then because of the discussion carried out above

$$\int_a^b | g' |^\alpha \, dx \geqslant \int_a^b | g_1' |^\alpha \, dx$$

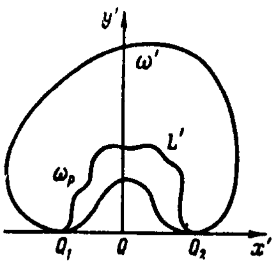

FIG. 6.

Furthermore, drawing a tangent to $f(x)$ at another point and replacing $g_1(x)$ on the corresponding portion by a linear function we obtain $g_2(x)$ and

$$\int_a^b | g_1'(x) |^\alpha \, dx \geqslant \int_a^b | g_2'(x) |^\alpha \, dx$$

Repeating this process without limit, we obtain the lemma.

Proof of theorem 2.7. The proof is by contradiction. Let P_1 and P_2 be two points of the connected portion of the inner boundary Γ' of the connected components of the flow domain ω', and let a point Q not belonging to ω' exist on the segment $[P_1, P_2]$. Then a whole interval of the segment $[P_1, P_2]$ with the endpoints Q_1, Q_2 lying on Γ' ($[Q_1, Q_2] \subseteq [P_1, P_2)$ belongs together with Q, to the domain $\omega \setminus \omega'$. Let a positive direction be selected on the boundary of the domain ω' (the domain ω' remains to the left) and let the arc $Q_1 Q^* Q_2$ be matched with the direction of travel, where $Q^* \in \Gamma'$ and Q^* does not belong to $[Q_1, Q_2]$. Let us introduce a new coordinate system by directing the Ox_1 axis along the segment $[P_1, P_2]$, and the Oy_1 axis perpendicularly to $[P_1, P_2]$ towards Q^*. We take the origin at Q.

Let $v(x_1, y_1)$ denote the function minimizing the functional (1.5) in the new variables. Let us consider the cylindrical surface $z = f(y_1)$ such that $f(0)$, $f(y_1) > 0$ for $y_1 > 0$, $f'(y_1) \geqslant 0$, $f''(y_1) \geqslant 0$. It can be shown that there exists such a cylindrical surface of the form mentioned, which intersects the surface $z = v(x_1, y_1)$ on the line L going from the point Q_1 to Q_2 with projection L' lying entirely within ω'. Let us consider the domain ω_p', which is an expansion of the domain ω' to the segment $[Q_1, Q_2]$ inclusive (Fig. 6). Let S be the domain bounded by a contour consisting of L' and the segment $[Q_1, Q_2]$. Let us put $v_0(x_1, y_1)$ equal to $v(x_1, y_1)$ in ω' and equal to zero in $\omega_p' \setminus \omega'$. Let us introduce the function

$$w(x_1, y_1) = \begin{cases} v(x_1, y_1) & \text{in } \omega' \setminus S \\ \max\{f(y_1), \quad v_0(x_1, y_1)\} & \text{in } S \end{cases} \tag{A.19}$$

in the domain $\omega_p{}'$.

Let us show that

$$\int\limits_{\omega_p{}'} \left\{\frac{\mu}{2} |\operatorname{grad} w|^2 + \tau_0 |\operatorname{grad} w| - cw\right\} d\omega \leqslant \int\limits_{\omega_p{}'} \left\{\frac{\mu}{2} |\operatorname{grad} v_0|^2 + \tau_0 |\operatorname{grad} v_0| - cv_0\right\} d\omega$$

To do this it is sufficient to establish that

$1^\circ \qquad w \geqslant v_0 \quad \text{in } \omega_p{}'$

$2^\circ \qquad \displaystyle\int\limits_{\omega_p{}'} |\operatorname{grad} w|^\alpha d\omega \geqslant \int\limits_{\omega_p{}'} |\operatorname{grad} v_0|^\beta d\omega, \qquad \alpha \geqslant 1$

The former is evident by virtue of the construction of the function w. To prove the latter inequality, let us note that the integrals should be considered only in the subdomain of the domain S where w and v_0 are different. Then, representing the double integral by iterated integrals, we obtain that 2^0 is equivalent to

$$\int\limits_a^b dx \sum_{i=1}^{\infty} \int\limits_{y_i(x)}^{y_{i+1}(x)} |f'(y)|^\alpha dy \leqslant \int\limits_a^b dx \sum_{i=1}^{\infty} \int\limits_{y_i(x)}^{y_{i+1}(x)} |\operatorname{grad} v_0|^\alpha dy \tag{A.20}$$

The inequality (A.20) follows from the inequality

$$\int\limits_a^b dx \sum_{i=1}^{\infty} \int\limits_{y_i(x)}^{y_{i+1}(x)} |f'(y)|^\alpha dy \leqslant \int\limits_a^b dx \sum_{i=1}^{\infty} \int\limits_{y_i(x)}^{y_{i+1}(x)} \left|\frac{\partial v_0}{\partial y}\right|^\alpha dy \tag{A.21}$$

The inequality (A.21) follows from lemma A.1 which asserts that

$$\int\limits_{y_i(x)}^{y_{i+1}(x)} |f'(y)|^\alpha dy \leqslant \int\limits_{y_i(x)}^{y_{i+1}(x)} \left|\frac{\partial v_0}{\partial y}\right|^\alpha dy$$

Thus the inequality (A.19) has been proved. Let us continue $w(x_1, y_1)$ from $\omega_p{}'$ to zero in the domain $\omega \setminus \omega_p{}'$ and let us consider the function

$$v^*(x_1, y_1) = \max\{v(x_1, y_1); w(x_1, y_1)\}$$

It is easy to see that $J(v^*) \leqslant J(v)$. The contradiction obtained proves theorem 2.7.

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